LETTER TO THE EDITOR

Levinson's theorem and non-local potentials

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Abstract. The modification of Levinson's theorem for the case of non-local potentials suggested by many authors is shown to be incorrect. The error is a result of overlooking a simple property of the inverse tangent function.

Except for the special case of a bound state at zero energy, Levinson's theorem for a local potential states that

$$\delta(0) - \delta(\infty) = n\pi \tag{1}$$

where $\delta(E)$ is the phase shift at energy E and n is the number of negative energy bound states. In the case of non-local potentials it has been suggested many times (Clement *et al* 1974, Englefield and Shoukry 1974 and references therein) that the theorem should be

$$\delta(0) - \delta(\infty) = (n + m)\pi \tag{2}$$

where m is the number of positive energy bound states. We show that this result cannot be valid in general by demonstrating that it is not true in a particular case. The source of the error in previous work can be traced to a property of the inverse tangent function, namely

$$t_1 = \tan^{-1}\left(\frac{(a-x)^2}{a-x}\right) \neq \tan^{-1}\left(\frac{a-x}{1}\right) = t_2$$

if $x > a$. In both cases $\tan t$ changes sign at $x = a$. However, we see that $\tan t = \sin t / \cos t$ has changed sign for different reasons, namely that in the first case the change of sign has arisen from the $\cos t_1$ term and in the second through the $\sin t_2$. The consequence of this is that $t_1 = t_2 + \pi$ for $x > a$, i.e. t_1 has a jump of π at $x = a$. The error that occurs in the derivation of equation (2) essentially is that t_1 is taken to be equal to t_2 for all x.

We now consider the particular example of Englefield and Shoukry. Equation (3.1) in their paper may be written, taking the local potential V_c to be equal to zero, as

$$u'' + k^2 u = g(r) \int_0^\infty g(s) u''(s) \, ds, \tag{3}$$

with

$$\int_0^\infty g^2(r) \, dr = 1. \tag{4}$$

Consider the particular case of the Saito non-local potential for which

$$g(r) = (6\alpha)^{1/2}\lambda(e^{-\alpha r} - 2e^{-2\alpha r}),$$

where α is a constant and λ is a switching-on parameter which is equal to 1 when (4) is satisfied. The phase shift for this potential is

$$\tan \delta = \frac{6x^{3/2}\lambda^2(2-x)^2}{(4+x)(1+x)[(4+x)(1+x)-4\lambda^2-7\lambda^2 x]+18x^2\lambda^2(2-x)} \tag{5}$$

where $x = k^2/\alpha^2$. The non-negative numerator is zero at $x = 2$. For λ less than, equal to or greater than 1, the denominator has a zero at x greater than, equal to or less than 2 respectively. The variation of δ with x is shown in figure 1 for the cases when λ is equal to 0·98, 1·00 and 1·02.
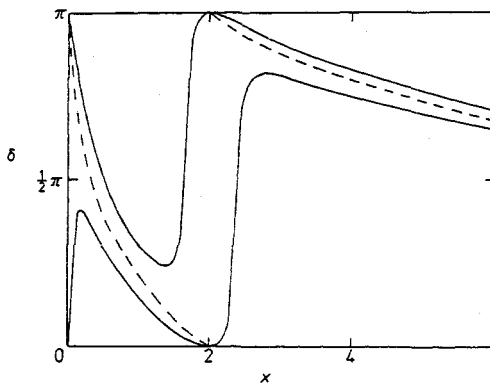

Figure 1. The phase shift (δ) as a function of energy ($x = k^2/\alpha^2$) for $\lambda = 0\cdot98$ (lower curve), 1·00 (broken curve) and 1·02 (upper curve).

As the potential is switched on a resonance appears and moves to $x = 2\cdot38$ for $\lambda = 0\cdot98$. With a further increase in λ the position of the resonance approaches $x = 2$ and its width decreases. At $x = 2$ the width is zero, i.e. there is a jump of π radians in the phase shift. A resonance of zero width means an infinite lifetime and thus a positive energy bound state. Further increases in λ increase the width of the resonance and continue to reduce its energy: for example, it occurs at $x = 1\cdot64$ for $\lambda = 1\cdot02$. We note also that although $\tan \delta = 0$ at $x = 2$, the phase shift is zero for $\lambda < 1$ but π for $\lambda > 1$.

For $\lambda = 0\cdot98$, there is almost a resonance at $x = 0\cdot2$. This becomes a reality at $\lambda = 0\cdot99$ with the resonance at $x = 0\cdot06$. Further increases force the resonance into a negative energy bound state which is reflected in the fact that $\delta(0) = \pi$.

We note that the use of the intrinsic function ATAN2 in the FORTRAN computer language for the calculation of the phase shift from equation (5) automatically gives the exact phase shift.

The example given in this letter is similar to the case of the Tabakin (1968) one-term separable potential discussed by Beam (1969) and Kermode *et al* (1975). In that case too, the phase shift jumps from 0 to π. A slightly different but related example of a positive energy bound state is mentioned by McKerrell and Kermode (1976). There, *both* sin δ and cos δ change sign and the phase shift shows a jump from $-\frac{3}{4}\pi$ to $\frac{1}{4}\pi$.

We conclude that at a positive energy bound state the phase shift shows a jump of π and suggest that Levinson's theorem is $\delta(0) - \delta(\infty) = n\pi$, where n is the number of negative energy bound states. This result is physically more satisfying than those for which π is added to the phase shift when $x < a$. First, the potential can be switched on smoothly. Second, the positive energy bound state is clearly seen at its correct energy. Finally, the positive energy bound state may be regarded as a resonant state with an infinite lifetime. Previous statements of Levinson's theorem for non-local potentials are unsatisfactory in that they are different for systems which have states with lifetimes of one million years and those with states that live for ever.

References

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